Axial Precession and Angle of a Spinning Disc

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Abstract
The relationship between the angle of a spinning glass disc and its rate of axial precession was investigated. The spinning disc was recorded with a high-speed camera and the angle and rate of precession of the disc determined using video analysis. The behaviour of the disc was compared with a theoretical model. It was found that the rate of axial precession of the real disc deviated slightly from the predictions of the theoretical model.

Keywords: Euler's Disc, spinning disc, axial precession, disc angle

I. INTRODUCTION

Spin a coin on a table. It moves with a spinning, rolling motion, getting lower and lower while spinning faster and faster until it stops. Another example of this motion is Euler's Disc, marketed by Joseph Bendik as a scientific-educational toy. Bendik named the toy after Leonhard Euler, who studied the relevant theory in the 18th century.[1] Euler's Disc consists of a heavy metal disk and a mirror-like base approximately three times the diameter of the disk, shown in Figure 1. An Euler's Disc moves just like a spinning coin, but spins for a much longer time. While the duration of spinning is theoretically mass independent,[2] the Euler's Disc spins for much longer due to the combination of high mass and low friction.

When a disc spins on a flat surface, it exhibits a rolling motion. This is illustrated in Figure 2, where the rolling motion of a disc of radius, r, can be seen as a combination of an axial precession, w_p, and an azimuthal rotation, w_s. As the disc spins it loses energy and there will be a decrease in the angle (θ) the disc forms with the surface, with a corresponding increase in the disc's rate of axial precession as its azimuthal rotation gradually decreases.[4]

Figure 1. A spinning Euler's Disk.[3]

A mathematical model, derived by Milan Batista, which assumes an ideal disc with negligible thickness and friction, gives the relationship between axial precession and the disc angle[5] as:

$$\omega_p = \sqrt{\frac{4g}{r sin\theta}} \tag{1}$$

The equation can rearranged in terms of the period, T, as:

$$T^2 = \frac{\pi^2 r sin\theta}{g} \tag{2}$$

Here we investigate whether Batista's theoretical model can be applied to a real spinning disc with non-negligible thickness and friction.

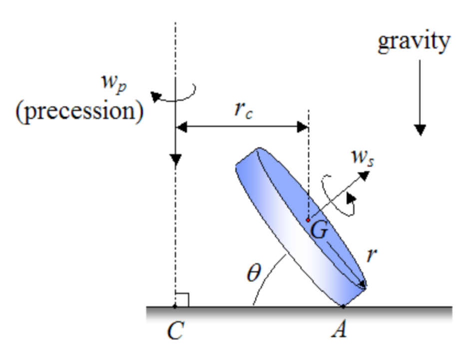

Figure 2. A disk spinning on a flat surface.[6]

II. METHODS

A 60 mm diameter, 5 mm thick glass disc, shown in Figure 3, was spun on a smooth granite surface. A video camera, recording at 240 fps, was placed horizontally at the level of the surface, so that the angle of the disc could be accurately measured. The disc was spun starting on its edge and recorded until it stopped spinning. Four complete spins were recorded and analysed using the Tracker video analysis program.

For each rotation, the angle (θ in Figure 2) was measured when the disc was edge-on to the camera, as shown in Figure 4. The time for one complete rotation was measured for each rotation, along with the average angle for that rotation. This was done for each rotation until the disc stopped spinning. Angle and period were measured for each of the four recorded spins. The spinning motion of the disc was unstable for angles above 50˚ and below 5˚, so the data was only analysed for angles within this range.

III. RESULTS AND DISCUSSION

The relationship between the rate of axial precession and the angle the disc forms with the surface is shown in Figure 5. As can be seen, as the disc angle decreases, the rate of angular precession

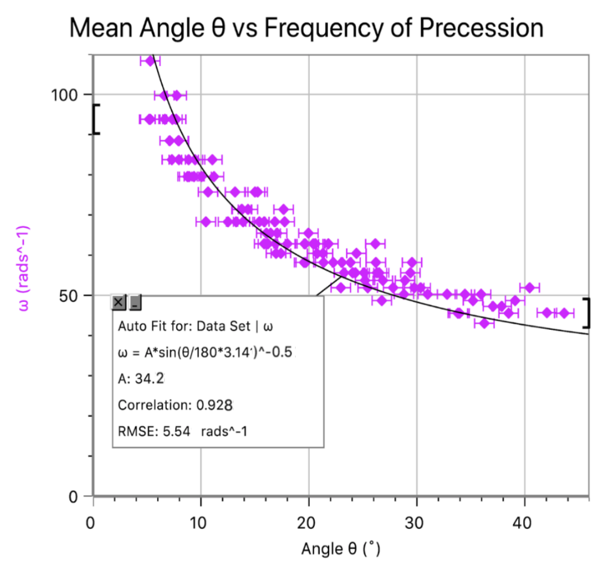

Figure 5. Mean Angle (θ) vs the Rate of Axial Precession with Equation 1 fit to the data.

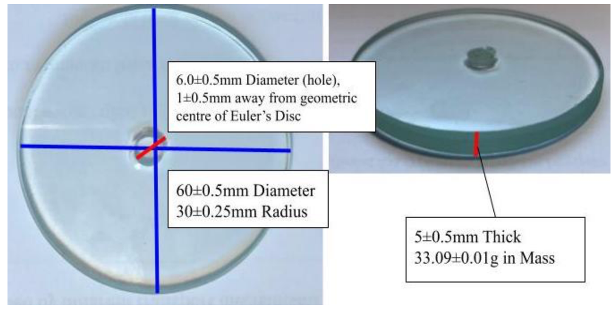

Figure 3. showing the diameter, radius, thickness and mass of the Euler's Disk used for this experiment.

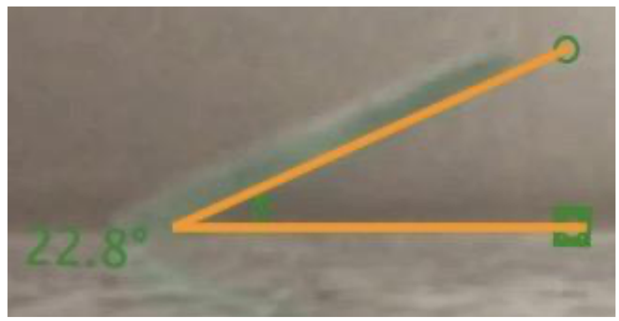

Figure 4: a sample from video analysis. The two orange lines show the angle formed by the plane of the disk and the base.

increases. According to Batista's Equation, an inverse root relationship is predicted between $sin\theta$ and w_p.

The fit of Bautista's model to the data gives:

$$w_p = 34.2s^{-1} \times (sin\theta)^{-0.5} \qquad (3)$$

It is clear that while the results for the glass disc follow the general trend predicted by Bautista's model, it is not a perfect fit. At high angles, the angular frequency is higher than predicted, and at low angles, it is lower. It is likely that the deviation of the results for the disc used here from the theoretical model is due to slipping[7] and the non-zero thickness of the disc. Further research is needed to confirm this.

The theoretical model, represented as Equation 2, suggests a proportional relationship between $sin(\theta)$ and the period of precession squared (T^2) with a proportionality constant of $\frac{\pi^2 r}{g}$. For the disc used

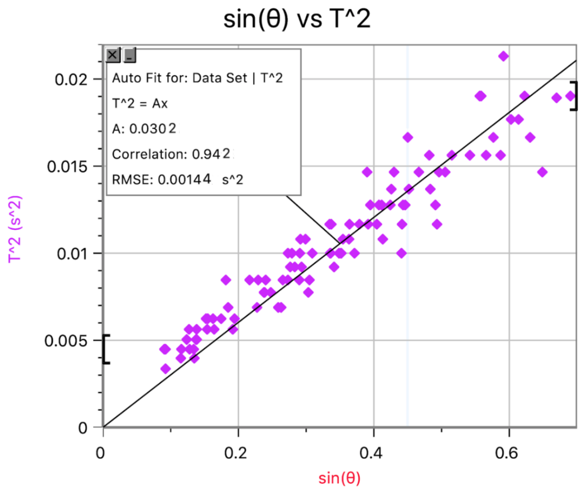

Figure 6. Presenting the results showing the positive proportional relationship between $sin(\theta)$ and T^2.

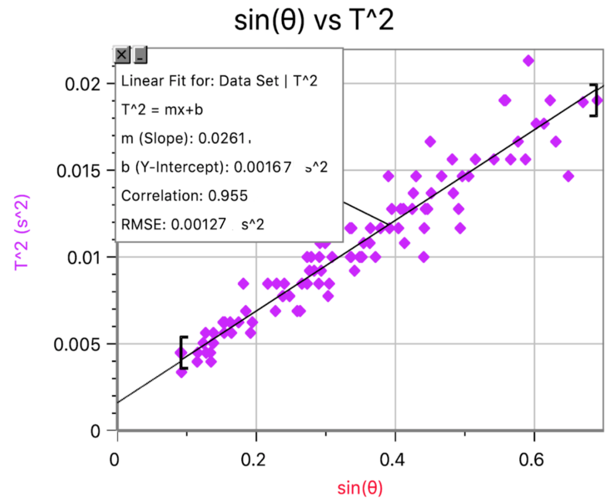

Figure 7 A linear relationship gives a better fit to the results in figure 5.

here, this constant has a value of $0.0303 \pm 0.0003\ s^2$. Graphing the results as T^2 vs $sin(\theta)$, as in Figure 6, gives a proportionality constant of $0.0302\ s^2$, which is only 0.4 % less than predicted. It is clear from Figure 6 that a proportional fit is not the best fit, as the period is greater than predicted for low angles and less than predicted for high angles. So while it is clear that the behaviour of the real disc does not follow theory, the deviation is very small. Figure 6 also shows that the periods become less consistent at larger angles, confirming that the precession of the real Euler's Disc is less stable at larger angles.

A linear relationship is fit to the results, as in Figure 7, but clearly does not pass through the origin, and has a slope that is 20% below the theoretically predicted value. However, it has a higher correlation than the proportional fit, hence is a more accurate empirical model for the behavior of the disc within the range of angles tested here. Thus, the empirical relationship between the period of axial precession as a function of disc angle for the disc used here can be expressed as:

$$T^2 = 0.0261\ s^2 \cdot sin(\theta) + 0.00167\ s^2 \quad (4)$$

According to theory, w_p is predicted to approach infinity as the disc angle approaches zero. In reality, the limitations of the friction between the disc and the surface imply that the disc will no longer maintain a stable motion at very low angles. Theory also predicts that w_p will approach $\sqrt{\frac{4g}{r}}$ as the angle approaches 90˚. For a real disc, the spinning motion at angles above 50˚ became unstable, likely due to the effects of the non-zero thickness of the real disc. Further research is suggested to confirm if and how friction and the thickness of the disc affect the stability of the spinning motion and the deviation of the behaviour from theory.

IV. CONCLUSION

Batista's theoretical model gives a reasonably close approximation of the spinning behaviour of the real glass disc used in this investigation for disc angles ranging from 5˚ to 45˚. Within this range, the square of the period of axial precession is proportional to the disc angle. The disc behaviour deviated slightly from the theoretical predictions, with the actual period being greater than predicted for low angles, and less for high angles.

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