Hole Diameter and Sinking Time of Water Clocks

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Abstract

Water clocks measure time by controlling the flow of water into a container. Ancient Persians used the sinking time of Fenjaans, bowls with holes in the center, as a metric for time. The relationship between hole diameter and sinking time of cylindrical containers was investigated. A theoretical model was derived using Torricelli's Law and Archimedes' Principle. The experimental relationship was found by drilling holes with diameters ranging from 2 to 13 mm in the center of nearly cylindrical stainless-steel containers and measuring their sinking time. It was found that there was a proportional relationship between the sinking time and the inverse square of the hole diameter with a proportionality constant which matched, within uncertainties, the proportionality constant predicted by the theoretical model.

Keywords: water clock, Torricelli's Law, buoyancy, flow rate

I. INTRODUCTION

Water clocks were commonly used for time measurement around the world before the introduction of pendulum clocks. One of the earlier forms of a water clock was a bowl with a hole in the center of the base, known as a Fenjaan in Persian (figure 1). In ancient Persia, farmers used Fenjaans to ensure equitability in water distribution. The Fenjaans measured the time each farmer could open the irrigation channels to their fields.[1] Even today, water clocks are still in use. In southern Thailand, water clocks are used to time the rounds in traditional gamecock crowing competitions.

This paper aims to develop a theoretical model for the relationship between hole diameter and sinking time of cylindrical stainless-steel containers, so that sinking time can be predicted given hole diameter and bowl dimensions. Though empirical data showing a correlation between hole diameter in containers and sinking time has been published,[2] this study aims to quantify the relationship and compare the theoretical model with the experimental results.

Figure 1. A Fenjaan sinking in water[1].

Torricelli's law states that the relationship between the velocity (v) of water which flows through a hole of small diameter, and the height of fluid above the position of the hole (h) is expressed by the formula,[3]

$$v = \sqrt{2gh} \quad . \tag{1}$$

In the situation normally covered by Torricelli's law, the water is flowing out of a hole in a stationary container, while in the case of this investigation, the water flows into the container as it sinks down into the water. Water clocks are effectively an inverse application of Torricelli's law: the velocity of the water flowing into the container is predicted to be proportional to the root of the height difference between the water level inside and outside the container.

For a stable floating container (figure 2a), the upward buoyant force is equal to the weight of the fluid that the body displaces, according to

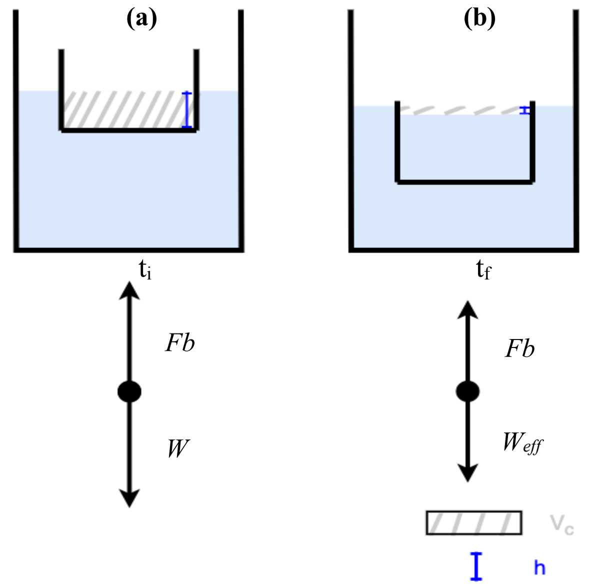

Figure 2. Diagrams for container sinking and simplified free body diagrams for t_i (a) and t_f (b).

Archimedes' principle.[4] For this investigation, the buoyant force and the weight acting on the container at the time when it is first placed in the water (t_i), and just before sinking (t_f), will be calculated using the assumption that there is no hole in the container. This approximation was made because the container sank at a slow rate where the acceleration was near zero. Under such an assumption, the weight (W) and buoyant force (F_b) for the sinking container at the beginning (t_i), shown in figure 2a, can be expressed as equation 2, where V_w and ρ_w are the volume and density of the container walls, A_c is the cross-sectional area of the container, ρ_f is the density of the fluid, and h is the initial difference in the water levels,

$$V_w \rho_w g = A_c h \rho_f g \, . \tag{2}$$

Rearranging the equation to solve for h, we obtain equation 3,

$$h = \frac{V_w \rho_w}{A_c \rho_f} \, . \tag{3}$$

When the container is almost submerged (figure 2b), the buoyant force on the walls of the container must be considered to determine the effective weight (W_{eff}) of the container. As most of the container is submerged in water, the weight of water displaced by the walls causes a buoyant force on the walls and decreases the effective weight. This gives equation 4 for the effective weight and buoyant force for the final time just before sinking,

$$V_w \left(\rho_w - \rho_f\right) g = A_c h \rho_f g \, , \tag{4}$$

and equation 5 for the final height of the container,

$$h = \frac{V_w\left(\rho_w - \rho_f\right)}{A_c \rho_f} \, . \tag{5}$$

The buoyant force decreases as more water fills the container, so the equation for the height, given a certain depth of water in the container *(y)* can be expressed as equation 6, where h_c is the total height of container.

$$h = \frac{V_w\left(\rho_w - \rho_f \frac{y}{h_c}\right)}{A_c \rho_f} \, . \tag{6}$$

Since the instantaneous flow rate (Q) is the product of the area of the hole (A_h *or* $\frac{\pi}{4}d^2$) and velocity (v) but also the product of the area of the container (A_c) and the instantaneous rate of change of the water level in the container (dy_c/dt), equation 7 can be obtained,

$$Q = A_h v = \frac{\pi}{4} d^2 \sqrt{2gh} = A_c \frac{dy_c}{dt} \, . \tag{7}$$

Rearranging equation 7 in terms of an integral of dy_c/dt and substituting h from equation 6, gives

$$\int_0^t dt = \frac{A_c}{A_h} \sqrt{\frac{A_c \rho_f}{2 V_w g}} \int_0^{h_c} \frac{1}{\sqrt{\rho_w - y \frac{\rho_w}{h_c}}} \, dy_c \, . \tag{8}$$

Solving the definite integral from $y = 0$ to $y = h_c$, gives,

$$t = \frac{8 A_c^{3/2} h_c}{\pi \sqrt{2 V_w g}} \left(\sqrt{\frac{\rho_w}{\rho_f}} - \sqrt{\frac{\rho_w - \rho_f}{\rho_f}}\right) \times d^{-2} \, . \tag{9}$$

By substituting the values for the measurements of the container and the fluid used for this investigation into equation 9, the proportionality constant between the sinking time and the hole diameter for the tested container was determined to be,

$$t = (1.02 \times 10^{-3} \, m^2 s) d^{-2} \, . \tag{10}$$

While a proportional relationship between sinking time and the squared reciprocal of hole diameter is predicted, it is expected that there will be a threshold value for very small hole diameters where water can no longer flow through the hole due to adhesion and surface tension, the model will then no longer be valid. In addition, viscous drag and the resulting turbulence in the fluid flow is expected to decrease the effective flow rate and increase the sinking time of the container systematically, compared to the times predicted by the model.[5]

II. METHODS

Six containers with dimensions shown in figure 3 and mass of 83 ± 3 g were drilled with holes with diameters varying from 2 mm to 13 mm. Then, the sinking time of each of those containers was recorded six times. The containers were dried after each trial. The diameter of the container was measured at relevant points and the weighted average diameter was determined and used when calculating the proportionality constant in equation 10. The density value of the stainless steel for the proportionality constant was obtained from the Engineering Toolbox.[6] Measurement uncertainty was mainly affected by the timing uncertainty and centering of the drilled holes. Uncertainty in timing was estimated by determining the variability in multiple trials, while uncertainty due to errors in hole size and centering was estimated by drilling a 5 mm hole in three separate containers and determining variability in sinking time of the three.

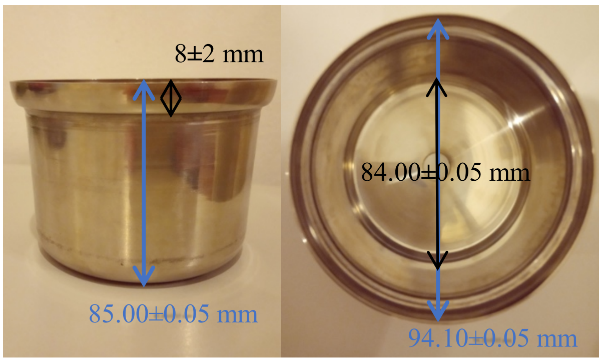

Figure 3. Dimensions of the container used. Container walls were 2.00 ± 0.05 mm thick.

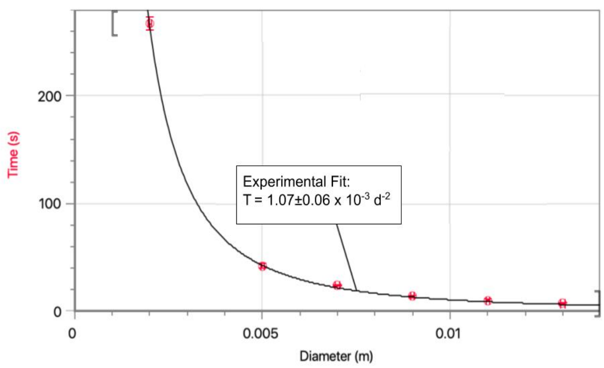

Figure 4. Empirical fit for the relationship between sinking time and hole diameter.

III. RESULTS AND DISCUSSION

Figure 4 and figure 5 show that hole diameter (d) and sinking time (t) for the container tested has the relationship,

$$t = (1.07 \pm 0.06 \times 10^{-3}\, m^2 s)\, d^{-2} \quad . \qquad (11)$$

As shown in figure 5, the experimental fit is consistent with the theoretical model within uncertainties, with only a 5% difference between the experimental data and the theoretically predicted constant. This confirms the validity of the assumptions that energy loss due to turbulence and viscous drag were negligible, and that the hole diameters tested were in the range where Torricelli's law is applicable.

Bowls with larger holes consistently took slightly longer to sink compared to the derived theoretical model than the smaller holes, as shown in figure 6. The gradient generated from the first 4 points deviated largely, with a gradient 12% higher than the original and 18% higher than the theoretical gradient, showing that larger holes generally took longer to sink compared to smaller holes. It is possible that for larger holes, the assumption of negligible acceleration during sinking was invalid, causing the model's predictions to be too low. Another possible explanation for the greater deviation from the model is that turbulence and viscosity effects had a more significant impact at larger hole diameters, although more work must be

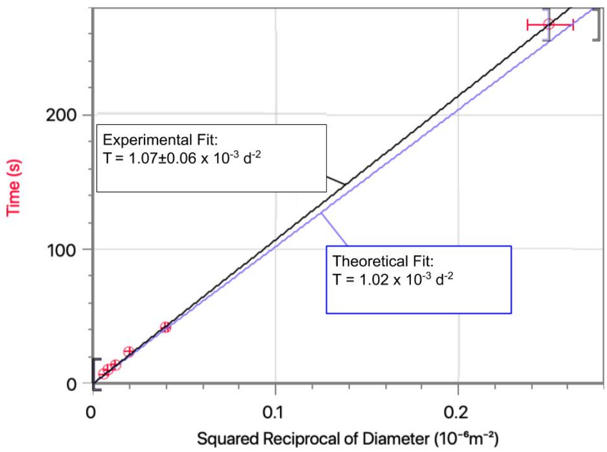

Figure 5. Comparison between the empirical results and theoretical model for the relationship between sinking time and squared reciprocal of hole diameter.

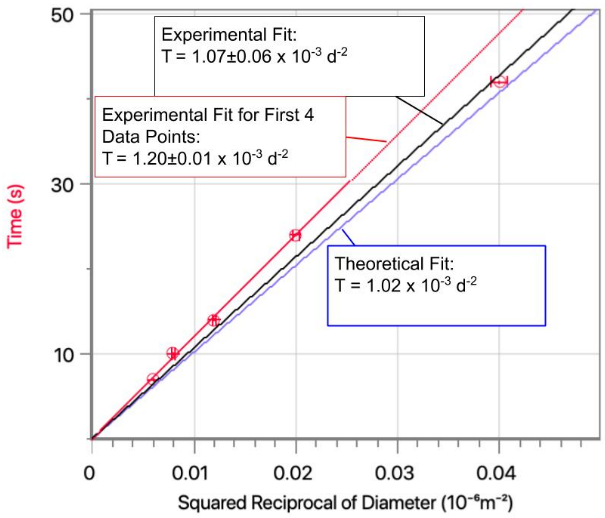

Figure 6. Figure zoomed to the first 5 data points. The red fit represents the gradient for first 4 data points.

done both theoretically and experimentally to confirm and explain this observation. The turbulence caused by the roughness in the drilled hole walls might have significantly increased energy losses and this would have increased the sinking time of the containers.

Another factor that might have caused a deviation was the inconsistency in locating the center of the container in the drilling process. Especially for the smaller diameter values, the container tilted to one side before the bottom of the container was fully covered with water, even causing minor oscillations at the beginning. As these factors were not considered in the derived model, it is possible that these factors caused the sinking time for the larger diameter holes to deviate from the theoretical model slightly more than the smaller holes.

The theoretical model derived here is shown to be valid for hole diameters from 2 mm to 13 mm. It is expected that viscosity is likely to be a factor increasing sinking time for very small holes. For relatively large holes the acceleration of the container may be non-negligible, rendering the model invalid. Further research is needed for values outside the range tested here, in order to construct a more comprehensive model for the relationship between hole diameter and sinking time.

IV. CONCLUSION

A theoretical model describing the relationship between the hole diameter and sinking time of a water clock was derived and tested for validity. It was shown that, for a nearly cylindrical metal container with dimensions shown in figure 1, and for hole diameters from 2 mm to 13 mm, there is a proportional relationship between inverse square of hole diameter and sinking time. The experimental proportional constant was shown to be 5% lower than the value predicted by the theoretical model, likely due to the unaccounted energy losses from viscous drag and turbulence as the water flows through the holes.

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